Kuwait University Math.101 August 11,1994

Mathematics Department Final Examination Duration: 2 Hours

Answer the following questions. Each question counts 10 points.

Calculators are not allowed

1. (a) Evaluate the following limit if it exists:

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}.$$

(b) Let
$$f(x)=\begin{cases}3+\sin Ax & if\ x\geq 0\\ x^2+2x+3, & if\ x<0.\end{cases}$$
Use the definition of the derivative to find A so that f is differentiable at $x=0$.

2. (a) Find an equation of the tangent line to the graph of the equation $\sqrt{xy}+\tan(1-x^2)=1$ at the point whose $x-$coordinate is 1.

(b) Use differentials to find a linear approximation for $\sqrt{7+\sqrt[3]{79}}$.

3. (a) A wire 20 cm long is to be cut into two pieces. If each piece is bent into the shape of a square. Where should the wire be cut so that the sum of their areas is minimum?

(b) Show that the function $f(x)=x+\dfrac{1}{x}$ satisfies the hypotheses of the mean value theorem on the interval $[1,2]$. Hence find the number c in $(1,2)$ that satisfies the conclusion of the theorem.

4. (a) Evaluate the following integrals:

(i) $\displaystyle\int\frac{\sin 3x}{(1-\cos 3x)^5}dx$, (ii) $\displaystyle\int_0^2\left|x^2+3x-4\right|\,dx$.

(b) Let $f(x)=\dfrac{1}{\sqrt{3x+1}}$. Find the number z in $(0,5)$ that satisfies the conclusion of the mean value theorem for definite integrals.

5. Let $f(x)=\dfrac{x^3-16}{x}$.

(a) Find the intervals on which f is increasing or decreasing, and find the local extrema, if any.

(b) Find the intervals on which f is concave upward or downward, and find the points of inflection, if any.

(c) Find the vertical and horizontal asymptotes for the graph of f, if any.

(d) Sketch the graph of the function. (*Hint* : $\sqrt[3]{2}\simeq 1.26$)

Good Luck

Model Answer: For

Math 101 (Final)

1. a) $\lim_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x} \cdot \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$ $\quad 2+2\sin x$ $\quad 2(1+\sin x)$

$= \lim_{x\to 0} \frac{(1+\sin x)-(1-\sin x)}{\tan x\left[\sqrt{1+\sin x}+\sqrt{1-\sin x}\right]} = 2\lim_{x\to 0} \frac{\sin x}{\tan x\,[\quad]}$

$= 2\lim_{x\to 0} \frac{1}{\cos x\,[\quad]} = \boxed{1}$

b) f is diff. at 1 if L.H.D = R.H.D at 0 ⇒

L.H.D at 0: $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{x^2+2x+3-3}{x} = \lim_{x\to 0^-} x+2 = 2$

R.H.D. at 0: $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{3+\sin Ax-3}{x} = A\lim_{x\to 0^+} \frac{\sin Ax}{Ax} = A$

$\Rightarrow \boxed{A=2}$

2. a) If $x=1 \Rightarrow \sqrt{y}=1 \Rightarrow y=1$, the point is $(1,1)$.

Using implicit Diff. ⇒

$\frac{xy'+y}{2\sqrt{xy}} + \sec^2(1-x^2)\cdot(-2x) = 0 \Rightarrow y'(1)=3$

The eq^n is $\quad y-1=3(x-1) \Rightarrow \boxed{y-3x+2=0}$

b) Let $f(x)=\sqrt{7+x^{1/3}}$. Take $x=8$, $\Delta x=-0.1$

$f(x+\Delta x) \simeq f(x)+f'(x)\cdot\Delta x \Rightarrow$, $f(8)=3$

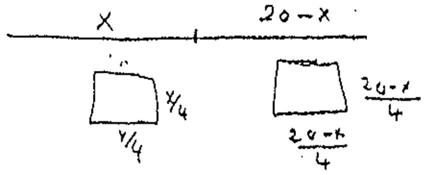
$f(7.9)=f(8-0.1) \simeq f(8)+f'(8)\cdot(-0.1)$, $f'(x)=\frac{x^{-2/3}}{3\cdot 2\sqrt{7+x^{1/3}}}$

$\simeq 3-\frac{0.1}{72} = 3-\frac{1}{720}$ $\qquad = \frac{1}{6\sqrt[3]{x^2}\sqrt{7+x^{1/3}}}$

$\simeq \frac{2160-1}{720} = \frac{2159}{720}$

$\simeq \boxed{2.998}$ $\qquad f'(8)=\frac{1}{6(4)3}=\frac{1}{72}$

3. a)



$A = (x/4)^2 + \left(\frac{20-x}{4}\right)^2 \Rightarrow D_A = [0, 20]$

$A'(x) = 2(x/4)\cdot 1/4 + 2\left(\frac{20-x}{4}\right)\cdot(-1/4) = 0 \Rightarrow x/4 - \left(\frac{20-x}{4}\right) = 0$

$\Rightarrow 2x - 20 = 0 \Rightarrow x = 10 \in D_A$, the only critical number

$A''(x) > 0 \Rightarrow A(10)$ is a local min. For min. should cut at $\boxed{x = 10}$ (mid point).

b) $f(x) = x + 1/x = \frac{x^2+1}{x}$ is a rational function; it is diff. on $(1,2)$ and conts on $[1,2]$.

But $f'(c) = \frac{f(2) - f(1)}{2-1} = 5/2 - 2 = 1/2$

or $1 - \frac{1}{c^2} = 1/2 \Rightarrow 1/c^2 = 1/2 \Rightarrow c = \pm\frac{1}{\sqrt{2}} \Rightarrow \boxed{c = \frac{1}{\sqrt{2}}}$ X

$c = \pm\sqrt{2}$

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4. a) (i) $I = \int \frac{\sin 3x}{(1-\cos 3x)^5}\,dx$, let $u = 1 - \cos 3x$
$du = 3\sin 3x\,dx$

$\Rightarrow I = \frac{1}{3}\int u^{-5}\,du = -\frac{1}{12}u^{-4} + C = -\frac{1}{12(1-\cos 3x)^4} + C$

(ii) $I = \int_0^2 |x^2 + 3x - 4|\,dx$.

$f(x) = |x^2 + 3x - 4| = |(x+4)(x-1)|$.

$x+4$: − − 0 + + + + + +
$x-1$: − − − − − − − − 0 + +
−4 0 1

$f(x) = \begin{cases} (x+4)(x-1), & \text{if } x \in (-\infty, -4] \cup [1, \infty) \\ -(x+4)(x-1), & \text{if } x \in (-4, 1) \end{cases}$

Hence

$$I = -\int_0^1 (x+4)(x-1)\,dx + \int_1^2 (x+4)(x-1)\,dx$$

$$= -\int_0^1 (x^2+3x-4)\,dx + \int_1^2 (x^2+3x-4)\,dx$$

$$= -\left[\frac{x^3}{3} + \frac{3}{2}x^2 - 4x \right]_0^1 + \left[\frac{x^3}{3} + \frac{3}{2}x^2 - 4x\right]_1^2$$

$$= \left(\frac{1}{3} + \frac{3}{2} - 1\right) + \left[\frac{8}{3} + \frac{3}{2}\cdot 4 - 8 - \left(\frac{1}{3} + \frac{3}{2} - 4\right)\right.$$

$$= -\frac{5}{6} + \frac{2}{3} - \frac{13}{6} = \boxed{5}$$

b) $$f(z) = \frac{1}{\sqrt{3z+1}} = \frac{1}{5}\int_0^5 \frac{dx}{\sqrt{3x+1}} \quad \text{But}$$

$$\int_0^5 \frac{dx}{\sqrt{3x+1}} = \frac{2}{3}\sqrt{3x+1}\Big]_0^5 = \frac{2}{3}(4-1) = 2$$

$$\Rightarrow \frac{1}{\sqrt{3z+1}} = \frac{2}{5} \Rightarrow$$

$$\Rightarrow \boxed{z = 7/4}$$

$D_f = \mathbb{R} \setminus \{0\}$.

$$f'(x) = \frac{x(3x^2)-(x^3-16)}{x^2} = \frac{2x^3+16}{x^2} \Rightarrow f'(x)=0 \Rightarrow x = -2$$
$$f' \text{ DNE} \Rightarrow x = 0$$
$$= \frac{2}{x^2}(x^3+16)$$

Interval	sign of f'	Variation of f
$(-\infty,-2)$	$-$	Dec. on $(-\infty,-2]$
$(-2,0)$	$+$	Inc. on $[-2,0)$
$(0,\infty)$	$+$	Inc. on $(0,\infty)$

$\Rightarrow f(-2)$ l. min $\Rightarrow f(-2) = \frac{-8-16}{-2} = 12$

Concavity:

$$f''(x) = 2\,\frac{x^2(3x^2)-(x^3+8)(2x)}{x^4} = 2\,\frac{3x^3-2x^3-16}{x^3}$$

$$= \frac{2}{x^3}(x^3-16) \Rightarrow f''(x)=0 \Rightarrow x = \sqrt[3]{16} = \sqrt[3]{8}\sqrt[3]{2} = 2(1.26) \approx 2.52$$

$f''(x)$ DNE $\Rightarrow x = 0$

Interval	sign of f''	Concavity
$(-\infty,0)$	$+$	CU
$(0,2.52)$	$-$	CD
$(2.52,\infty)$	$+$	CU

$P(2.52, f(2.52))$ is a point of inflection.
$P(2.52, 0)$

No horizontal asymptote; for vertical asymptotes

$\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to 0^-} f(x) = +\infty$; $x=0$ V.A. ; No symmetry;
x-intercept: $x = 2.52$